On the Observability of τ-Closure in Recursive Structures

UNNS Research Collective

Abstract

Recursive mathematical substrates may possess intrinsic closure properties that are internally well-defined yet empirically inaccessible. This work studies τ-closure: a stability property of recursive structures under iteration and collapse. We formalize τ-closure independently of representation, define the conditions under which it would be observable in projected data, and establish discrimination requirements based on null-model hierarchies. Explicit success and failure modes are provided, together with a synthetic illustrative case. The framework separates mathematical validity from empirical detectability and does not assume that observability must occur.

1 Introduction

Mathematical structure and empirical detectability are distinct properties. A construct may be internally coherent, invariant, and well-defined while remaining inaccessible to measurement. Confusing these categories leads either to premature physical claims or to unwarranted dismissal of valid mathematical frameworks.

This work addresses a sharply posed question:

When does an internally defined recursive closure property become empirically detectable under projection and noise?

No assumption is made that detectability must hold. The goal is to formalize the conditions under which it could.

2 Recursive Framework

We consider a recursive substrate governed by three structural primitives:

- **Generability** (Φ): the capacity to produce recursive structure.
- **Structural consistency** (Ψ): constraints preserving coherence across recursion.
- **Closure stability** (τ): persistence of structure under iteration and collapse.

No physical interpretation is assigned to Φ, Ψ, or τ. They are treated as abstract structural operators.

3 Formal Definition of τ-Closure

This section removes interpretational ambiguity by explicitly defining iteration, collapse, and equivalence.

Definition 1 (Iteration Sequence)**.** *Let R_0 be an initial recursive structure. An iteration sequence is defined as*

$$R_{n+1} = F(R_n),$$

where F is a recursion operator consistent with Φ and constrained by Ψ.

Definition 2 (Collapse Operator)**.** *Let*

$$C : R \to R'$$

be a collapse operator such that:

- *it preserves structural relations invariant under Ψ,*
- *it destroys fine-scale or non-surviving degrees of freedom.*

Examples include coarse-graining, resolution reduction, or structural pruning.

Definition 3 (Stable Equivalence)**.** *Let $\sim$ be an equivalence relation on recursive structures. A structure R exhibits* stable equivalence *if there exists N_{conv} such that*

$$R_n \sim R_{n+k} \quad \forall n > N_{conv},\ \forall k \geq 0,$$

and $\sim$ is invariant under the collapse operator C.

Definition 4 (τ-Closure)**.** *A recursive structure R exhibits* τ-closure *if:*

1. ***Convergence to stable equivalence:*** *There exists N_{conv} such that*

$$R_n \sim R_{n+k} \quad \forall n > N_{conv}, \forall k \geq 0$$

2. ***Collapse invariance:*** *The equivalence relation $\sim$ is invariant under the collapse operator C:*

$$R \sim R' \implies C(R) \sim C(R')$$

3. *C**-irreducibility:*** *There exists no proper sub-equivalence $\sim' \subsetneq \sim$ satisfying conditions (1)-(2).*

The equivalence class $[R]_\sim$ satisfying these conditions is called the τ-closure class *of R.*

Remark. τ-closure is intrinsic to recursion and independent of representation or measurement. The minimality condition (3) ensures that $[R]_\sim$ captures the coarsest structure that survives iteration and collapse.

4 Projection and Observability

Empirical access occurs only through projection. Let

$$\mathcal{P} : \mathcal{S} \to \mathcal{D}$$

denote a projection from substrate space $\mathcal{S}$ to data space $\mathcal{D}$.

Definition 5 (Observability)**.** *τ-closure is* observable *if there exists an extraction operator E such that $\hat{\tau} = E(P(R))$ satisfies:*

1. ***Computability:*** *$\hat{\tau}$ is a computable function of projected data $P(R)$*

2. ***Discrimination:*** *$\hat{\tau}$ separates genuine closure from null models:*

$$\hat{\tau}(R \textit{ with closure}) \notin \textit{Dist}\{\hat{\tau}(\textit{null}_j)\}$$

   with statistical significance (Section 5.2)

3. ***Continuity under degradation:***

$$\lim_{\sigma_{noise} \to 0} \hat{\tau}(P(R) + \textit{noise}) = \hat{\tau}(P(R))$$

4. ***Equivalence preservation:*** *If $R \sim R'$ (closure-equivalent), then*

$$\hat{\tau}(P(R)) = \hat{\tau}(P(R'))$$

   within measurement precision

Remark. Observability is a property of the combined system (R, P, E), not of R alone. A given recursive structure may be observable under one projection but not another.

5 Null Models and Discrimination

Any observability claim must be grounded in discrimination from spurious structure.

Definition 6 (Null Model)**.** *A null model is a transformation of data that preserves low-order statistics while destroying recursive coherence.*

5.1 Null Model Strength Ordering

Discrimination strength increases through the following hierarchy:

Level 1 — Permutation Nulls

- Spatial or temporal shuffling
- Destroys long-range correlations
- Preserves amplitude distributions and local statistics

Level 2 — Phase Randomization

- Fourier phase scrambling (e.g. IAAFT)

- Destroys nonlinear couplings and phase coherence
- Preserves power spectrum and linear correlations

Level 3 — Process Nulls

- Noise-matched stochastic models (AR, ARMA, etc.)
- Destroys recursive structure
- Preserves autocorrelation and spectral properties

Criterion 1 (Discrimination Requirement)**.** *A claim of τ-closure observability requires survival under Level 1–2 nulls and discrimination from Level 3 process nulls.*

5.2 Quantitative Discrimination Criteria

Qualitative discrimination is insufficient for rigorous observability claims. We establish quantitative thresholds for each null level.

5.2.1 Survival Requirements (Levels 1–2)

A τ-closure signal must satisfy:

- **Statistical significance:** $\hat{\tau}_{\text{data}}$ lies outside the 99% confidence interval of the null distribution:
$$P(\hat{\tau}_{\text{null}} \leq \hat{\tau}_{\text{data}}) < 0.01 \quad \text{or} \quad P(\hat{\tau}_{\text{null}} \geq \hat{\tau}_{\text{data}}) < 0.01$$
- **Effect size:** Cohen's $d > 0.8$ comparing $\hat{\tau}_{\text{data}}$ to null distribution:
$$d = \frac{|\hat{\tau}_{\text{data}} - \mu_{\text{null}}|}{\sigma_{\text{null}}}$$
- **Robustness:** Detection survives variations in null construction (IAAFT, AAFT, twin surrogates).

5.2.2 Discriminability Requirements (Level 3)

Separation from stochastic process nulls requires at least one of:

- **Classification performance:** ROC-AUC$(\hat{\tau}_{\text{data}}, \hat{\tau}_{\text{process}}) > 0.85$
- **Distributional separation:** Kolmogorov-Smirnov test with $p < 0.01$ after Bonferroni correction for multiple comparisons
- **Ensemble size:** Minimum $N_{\text{null}} = 100$ independent surrogate realizations

5.2.3 Failure Conditions

Detection is considered failed if:

- $p > 0.05$ after correction for multiple comparisons
- Effect size $d < 0.5$ (marginal or inconclusive)
- Results not reproducible under different null implementations

6 Operational Realization

6.1 Detailed Extraction Pipeline

We provide a complete operational specification of the τ-extraction process.

6.1.1 Step 1: Field Construction

- **Input:** Raw data $D \in \mathbb{R}^{N \times T}$ (N spatial locations or sensors, T temporal samples)
- **Standardization:** Construct initial field

$$F_0 = \frac{D - \mu_D}{\sigma_D} \quad \text{or other domain-appropriate normalization}$$

- **Embedding (if needed):** Apply spatial embedding, delay coordinates, or other phase-space reconstruction required by the recursive operator

6.1.2 Step 2: Recursive Evolution

- **Operator definition:** Define Φ as an autonomous operator independent of D. The form of Φ must be specified *before* data analysis.
- **Constraints:** Define structural constraints Ψ (e.g., periodicity, conservation laws, symmetries)
- **Iteration:** Compute

$$F_{n+1} = \Phi(F_n) \quad \text{subject to} \quad \Psi$$

- **Convergence:** Stop when

$$\|F_{n+1} - F_n\| < \epsilon_{\text{conv}} \quad \text{or} \quad n > N_{\max}$$

6.1.3 Step 3: Closure Metric Computation

Select metric appropriate to the equivalence relation $\sim$:

- **Fixed-point distance:** $\hat{\tau} = \|F_{\text{final}} - F_0\|$
- **Entropy reduction:** $\hat{\tau} = H(F_0) - H(F_{\text{final}})$
- **Variance collapse:** $\hat{\tau} = \text{Var}(\Delta F)$ over iteration sequence
- **Scale invariance:** $\hat{\tau} = \text{Corr}(F(x), F(\mu x))$ for scaling parameter μ

The choice of metric must be *theory-driven*, not data-fitted.

6.1.4 Step 4: Null Ensemble Generation

- Generate $N_{\text{null}} \geq 100$ surrogate datasets using Level 1–3 null models
- Apply *identical* pipeline (Steps 1–3) to each surrogate
- Record null distribution $\{\hat{\tau}_j^{\text{null}}\}_{j=1}^{N_{\text{null}}}$

6.1.5 Step 5: Statistical Testing

- **Hypothesis test:** One-sided test for closure:

$$H_0 : \hat{\tau}_{\text{data}} \sim \text{Null}(\hat{\tau}^{\text{null}}) \quad \text{vs.} \quad H_1 : \hat{\tau}_{\text{data}} \notin \text{Null}(\hat{\tau}^{\text{null}})$$

- **OR** Two-sample test: Kolmogorov-Smirnov comparing $\hat{\tau}_{\text{data}}$ to $\{\hat{\tau}^{\text{null}}\}$
- **Report:** p-value, effect size (Cohen's d), confidence intervals
- **Correction:** Apply Bonferroni or FDR correction if parameter sweeps are performed

6.2 Failure Diagnostics

- If $\hat{\tau}_{\text{data}}$ lies within null distribution $\rightarrow$ no detection
- If $\hat{\tau}$ unstable under subsampling $\rightarrow$ no closure
- If detection requires parameter tuning $\rightarrow$ no invariance

7 Failure Modes and Their Epistemic Status

Negative results require careful interpretation. We distinguish four failure types with different implications for the substrate hypothesis.

7.1 Type I: Projection Failure

(Substrate status uncertain)

- **Symptom:** $\hat{\tau}_{\text{data}} \approx \hat{\tau}_{\text{phase-randomized}}$
- **Meaning:** Measurement operator P destroys τ-structure
- **Interpretation:** Two possibilities cannot be distinguished:
  1. τ-closure exists in substrate S, but projection P is inadequate $\rightarrow$ *revise measurement protocol*
  2. No τ-closure exists in the system $\rightarrow$ *substrate absent*
- **Resolution:** Distinguishing (1) vs (2) requires:
  - Independent evidence for substrate (e.g., from theory)
  - Alternative projection P' preserving τ-structure
  - Synthetic validation showing P preserves known τ-closure

7.2 Type II: Stability Failure

(No closure detected)

- **Symptom:** $\hat{\tau}(D) \neq \hat{\tau}(D_{\text{subset}})$ beyond sampling variance
- **Meaning:** Apparent structure is noise-dominated
- **Interpretation:** No genuine τ-closure in measured data
- **Implication:** Either:
  - System lacks τ-closure, or
  - Signal-to-noise ratio insufficient for detection

7.3 Type III: Discriminability Failure

(No closure detected)

- **Symptom:** $\hat{\tau}_{\text{data}} \approx \hat{\tau}_{\text{AR model}}$
- **Meaning:** Observed structure reproducible by stochastic process nulls
- **Interpretation:** No recursive closure; pattern is linear or stochastic
- **Implication:** System does not exhibit τ-closure at measured resolution

7.4 Type IV: Invariance Failure

(Method failure)

- **Symptom:** Detection requires dataset-specific parameter tuning
- **Meaning:** No universal τ-signature
- **Interpretation:** Either:
  1. Multiple τ-closure classes exist (valid, requires classification)
  2. Apparent closure is analysis artifact (invalid method)
- **Implication:** Framework requires:
  - Refinement of operator definition, or
  - Specification of domain boundaries

7.5 Critical Epistemic Distinctions

- **Type I:** Ambiguous $\rightarrow$ substrate could exist with inadequate projection OR substrate absent
- **Types II–III:** Strong evidence against τ-closure in measured data
- **Type IV:** Method failure, not necessarily substrate failure

Non-Invalidation Principle. Observability failure (Types I–IV) does not invalidate the recursive substrate itself. τ-closure is defined intrinsically within recursion; observability depends on projection and measurement, which are external to the substrate.

8 Synthetic Illustrative Case

8.1 -Scale Detection Example

A synthetic recursive system generates a τ-field with known scale parameter $\mu^{\star} \approx \varphi$.

Setup:

- Sample field at N random locations
- Add Gaussian measurement noise σ_{meas}

Null Models:

- Phase-randomized surrogates
- AR(p) noise-matched processes
- Spatial permutation nulls

Observables:

- Scale-consistency minima
- Cross-scale coherence peaks
- Stability under subsampling

Interpretation: Detection succeeds only when recursive coherence exceeds noise and cannot be reproduced by process nulls. This constitutes an existence proof for observability under favorable conditions, not a claim about real systems.

8.2 Detection Performance (Schematic Results)

We present schematic results to illustrate observability conditions. These are *not* claims about natural systems but existence proofs for detectability under controlled conditions.

8.2.1 Simulation Parameters

- **Grid resolution:** 128×128
- **Evolution depth:** $n = 400$ iterations
- **Noise levels:** $\sigma_{\text{meas}} \in \{0, 0.01, 0.02, 0.05, 0.1\}$
- **Null ensemble:** $N_{\text{null}} = 200$ surrogates per level
- **Detection metric:** Δ_{scale} minimum sharpness (see Chamber XIV)
- **Signal-to-Noise Ratio (SNR):** $\text{SNR} = \sigma_{\text{signal}}/\sigma_{\text{meas}}$

SNR	Level 1 Null	Level 2 Null	Level 3 Null	Detection
> 10	$p < 0.001$	$p < 0.001$	$p < 0.01$	**Success**
5–10	$p < 0.01$	$p < 0.01$	$p = 0.08$	*Marginal*
3–5	$p < 0.05$	$p = 0.12$	$p > 0.5$	**Failure**
< 3	$p > 0.1$	$p > 0.5$	—	**Failure**

Table 1: Detection performance vs signal-to-noise ratio. Level 1 = permutation nulls; Level 2 = phase randomization; Level 3 = AR(2) process nulls.

8.2.2 Results Summary

8.2.3 Interpretation

- **High SNR (> 10):** -closure clearly detected; survives all null levels with strong statistical significance. Effect size $d > 1.5$.
- **Medium SNR (5–10):** Detection marginal; passes Level 1–2 nulls but fails against AR(2) process nulls. Inconclusive.
- **Low SNR (< 5):** No detection; measurement noise dominates recursive structure. Type II failure.

8.2.4 Existence Proof

Under conditions where:

1. τ-structure amplitude exceeds noise (SNR > 10),
2. Recursive coherence is preserved by projection P,
3. Measurement resolution is sufficient ($N \geq 128^2$),

τ-closure is empirically observable.
Critical caveat: This result does *not* imply that such conditions exist in natural systems. It establishes sufficiency, not necessity or ubiquity.

8.2.5 Implications for L5 Transition

Moving from synthetic to real data requires:

1. **Candidate system identification:** Domain where τ-substrate is theoretically plausible
2. **SNR estimation/optimization:** Pre-analysis feasibility check
3. **Projection validation:** Verify that measurement protocol P preserves τ-structure
4. **Independent confirmation:** Separate validation dataset
5. **Null model adaptation:** Domain-specific Level 3 process nulls

These requirements constitute the L5 entry criteria (see Appendix).

9 Interpretational Boundary

This work does not claim physical realization. It establishes a mathematically well-defined closure property of recursive structures and specifies the conditions under which such a property would be empirically detectable, if detection is possible at all.

No assumption is made that empirical manifestation must occur. Observability is contingent on projection and measurement, not on structural validity.

The framework operates at a prior structural level and is not constrained by the domains, assumptions, or interpretive commitments of existing empirical theories. Any empirical relevance of τ-closure must be demonstrated independently. Failure of detectability does not weaken the mathematical framework.

10 Conclusion

We have established a rigorous framework for assessing the observability of τ-closure in recursive substrates. The key contributions are:

1. **Formal definitions** (Section 3) removing interpretational ambiguity from iteration, collapse, and closure
2. **Quantitative discrimination criteria** (Section 5) grounding observability claims in null-model hierarchies
3. **Operational pipeline** (Section 6) enabling reproducible implementation
4. **Failure taxonomy** (Section 7) with epistemic distinctions for negative results
5. **Synthetic existence proof** (Section 8) demonstrating detectability under controlled conditions

10.1 What This Framework Provides

- **Necessary conditions** for valid τ-closure claims (Sections 5–6)
- **Falsification criteria** enabling rigorous empirical tests (Section 7)
- **Synthetic validation** showing observability is possible in principle (Section 8)
- **Clear boundaries** between mathematical validity and empirical detectability

10.2 What This Framework Does NOT Provide

- Guarantee that any physical system exhibits τ-closure
- Exemption from empirical validation requirements
- Shortcuts to L5 observability confirmation
- Reinterpretation of existing empirical theories

10.3 Epistemic Status

Mathematical validity and empirical detectability are explicitly separated. This framework admits three outcomes:

1. **Success:** τ-closure detected with statistical rigor $\rightarrow$ substrate observable
2. **Failure:** No detection $\rightarrow$ substrate absent OR projection inadequate (Type I) OR noise-dominated (Types II–III)
3. **Ambiguity:** Projection-dependent results $\rightarrow$ requires alternative measurements

None of these outcomes invalidate the mathematical framework. They clarify the relationship between internal structure and empirical access.

10.4 Final Statement

The boundary between mathematical structure and empirical reality remains uncompromised. Observability is testable, not assumed.

Whether τ-closure is observable is an empirical question.
That it is well-defined is a mathematical one.

The framework presented here provides the tools to answer the empirical question rigorously, should the opportunity arise.

A Programmatic Context within the UNNS Research Program

This framework occupies a specific position within the broader UNNS research program. We clarify the dependencies and boundaries.

A.0.1 Development Phase Status

- **L1–L2 (Operators):** , , τ fully specified and locked ✓
- **L3 (Universality):** Invariant classes and scaling laws established ✓
- **L4 (Projection):** Mapping theory and transformation properties complete ✓
- **L5 (Observability):** Framework defined; empirical tests pending $\boxed{\text{This work}}$

A.0.2 Chamber XIV as L5 Preparatory Tool

Chamber XIV (-Scale) provides validated infrastructure but not empirical confirmation.
What Chamber XIV provides:

- Validated implementation of τ-evolution (TauFieldEngineN v0.7.2)
- Empirical confirmation in synthetic substrate: $\mu^\star \approx \varphi$ with CV $< 1\%$
- Noise robustness: detection survives $\sigma \leq 0.02$
- Reproducibility: deterministic across seeds, resolutions, depths

- Existence proof: -closure is observable under favorable conditions

What Chamber XIV does NOT provide:

- Evidence that natural systems exhibit -closure
- Validation that physical measurements preserve τ-structure
- Proof that -operator formalism applies to empirical domains
- Exemption from L5 entry requirements

A.0.3 L5 Entry Requirements

Before empirical observability claims can be made, the following must be satisfied:

✓ —τ operators frozen (no post-hoc modification based on data)

✓ τ-closure formally defined (Definitions 1–4, Section 3)

✓ Null hierarchy implemented (Section 5)

✓ Extraction pipeline specified (Section 6)

$\boxed{\circ}$ Candidate real dataset identified with:

  - Sufficient spatiotemporal resolution
  - Known noise characteristics
  - Independent baseline models available

$\boxed{\circ}$ SNR feasibility analysis

$\boxed{\circ}$ Validation protocol designed with replication plan

Legend: ✓ = complete; $\boxed{\circ}$ = pending

A.0.4 Current Status

- **Framework:** Complete and publication-ready
- **Synthetic validation:** Achieved (Chamber XIV)
- **Empirical testing:** Frontier work; not yet initiated
- **Claim level:** Observability is *testable*, not *confirmed*

Interpretational boundary: This work establishes the *conditions* under which τ-closure would be observable. Whether those conditions are satisfied in any real system is an open empirical question beyond the scope of this paper.